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Numerical Simulation of Non-Stationary Heat Conduction in Axisymmetric Bodies

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***Abstract*—** The distribution of the temperature field of elements of three-dimensional structures is examined in the article and the influence of non-homogeneities on the temperature field in three-dimensional axisymmetric structures is studied. Based on the variational formulation of the problem, a finite element model for solving axisymmetric problems of heat distribution in three-dimensional bodies was built based on the finite element method; a computational algorithm of the solution and the software were implemented. Shape functions corresponding to linear triangular elements are used in the finite element solution to the problem. They ensure its continuity at the boundary with neighboring elements since this distribution is linear along any side of the triangle and with the same change in value at the nodes, the same changes will occur along the entire inner boundary. A discrete finite element mesh model is a set of parameters that consist of the total number of finite elements and mesh nodes, arrays of their node coordinates, and node numbers of finite elements. To verify the reliability of the results obtained, a computational experiment was conducted, related to the study of the effect of the increase in the number of finite elements on the convergence of solutions. Analysis of the experimental results confirms the numerical convergence of temperature values as the finite element mesh is refined. At the initial stage of the study, the influence of temperature distribution for homogeneous structural elements was studied.

***Keywords*—** *thermal conductivity, non-stationarity, cylindrical coordinates, discretization, temperature, finite element (FE) method*

1. INTRODUCTION

The article develops a numerical model for solving time-dependent heat conduction problems in axisymmetric bodies; provides a solution algorithm; studies the influence of geometric characteristics on the temperature field for inhomogeneous axisymmetric bodies, and analyzes the heat flow in a cylindrical space with a rectangular or circular cross-section.

2. RELATED WORKS

Many important physical three-dimensional problems can be solved using two-dimensional elements. An example of such a solution is the problem of radial heat transfer through concentric cylinders with varying heat conduction coefficients. The heat flow propagates radially and along the axis [1] at a

sufficiently large cylinder length. Monograph [2] describes a general approach to solving heat conduction problems and presents systematic instructions for creating calculation models. In [3], several approaches for solving a time-dependent heat conduction equation in an axisymmetric multilayer medium are explored. The problem is solved by combining the Fourier method with the matrix method. Solutions to the first and third boundary value problems are discussed. In reference [4], the authors proposed an automated system for computer modeling and simulation. This system not only enables the solution of mathematical models but also allows for the integration of data from real-world objects, enhancing the models by introducing correction factors. Article [5] describes computational results of numerical modeling of an axisymmetric body of an optimized shape designed to minimize aerodynamic drag force for heat dissipation in a convective gas flow. The research in [6] introduces a method for calculating the time-dependent

temperature distribution in a metallic body, supported by illustrative examples.

3. STATEMENT OF THE PROBLEM

The axisymmetric non-stationary heat transfer in a system of cylindrical coordinates (r,z,θ) is determined by the following differential equation [2, 6]:

$$K_{rr}\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}K_{rr}\frac{\partial T}{\partial r}+\frac{K_{\theta\theta}}{r^2}\frac{\partial^2 T}{\partial \theta^2}+K_{zz}\frac{\partial^2 T}{\partial z^2}+Q=\rho c\frac{\partial T}{\partial t} \quad (1)$$

where

$T=T(r,z,\theta,t)$ is the temperature field;

$K_{rr},K_{\theta\theta},K_{zz}$ are the thermal conductivity coefficients in the corresponding directions;

$Q=Q(r,z,\theta,t)$ is the power of heat sources inside the material;

ρ is the density of the material;

c is the heat capacity of the material;

r is the distance from the axis of symmetry to the element;

θ is the azimuthal angle.

If a three-dimensional body has geometric symmetry about the Oz - axis, then this body is called an axisymmetric body. If, in addition, the physical quantity under study does not depend on θ, then differential equation (1) is reduced to the following relationship:

$$K_{rr}\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}K_{rr}\frac{\partial T}{\partial r}+K_{zz}\frac{\partial^2 T}{\partial z^2}+Q=\rho c\frac{\partial T}{\partial t}. \quad (2)$$

Three types of boundary conditions are considered [7]:

T_{s_1} - temperature on the surface of body S_1:

$$T_{s_1}=T(r,z,t),\quad r,z\in S_1, \quad (3)$$

q - heat flux through the surface of body S_2:

$$K_{rr}\frac{\partial T}{\partial r}\ell_r+K_{zz}\frac{\partial T}{\partial z}\ell_z=-q, \quad (4)$$

where ℓ_r,ℓ_z - are the direction cosines of the normal vector to surface S_2.

convective heat exchange on the surface of body S_3:

$$K_{rr}\frac{\partial T}{\partial r}\ell_r+K_{zz}\frac{\partial T}{\partial z}\ell_z=-h(T_s-T_b), \quad (5)$$

where h is the coefficient of heat exchange with the external medium, T_s is the temperature (unknown) on the surface S_3, T_b is the ambient temperature.

The temperature of the body at initial time t_0 is $T_0=T(r,z,t_0)$. If surface S_2 is isolated, then $q=0$.

In this case, the solution to the system consisting of equation (1) and boundary conditions (3)-(5) will lead to the minimization of the functional [8-12]:

$$\Phi(r,z)=\int_V \frac{1}{2}\left[rK_{rr}\left(\frac{\partial T}{\partial r}\right)^2 + rK_{zz}\left(\frac{\partial T}{\partial z}\right)^2 - 2rQT + 2\lambda\frac{\partial T}{\partial t}T \right]dV + \int_{S_1} qTdS + \int_{S_2}\frac{h}{2}\left(T-T_b\right)^2 dS \quad (6)$$

where V is the volume; q is the heat flux on the given surface S_1; S_2 is the surface on which convective heat exchange occurs.

3. SOLUTION METHOD

When solving the problem, the axisymmetric domain is partitioned into finite elements, generally taken in the form of triangles (Fig. 1). All subsequent formulas are given for triangular finite elements. The functional is written as the sum of all finite elements. Thus, (6) takes the following form:

$$\Phi(r,z)=\sum_{e=1}^{n}\left[\frac{1}{2}\cdot\int_{V^{(e)}}\left\{g^{(e)}\right\}^T\cdot\left[D^{(e)}\right]\cdot\left\{g^{(e)}\right\}dV-\int_{V^{(e)}}\left(rQ^{(e)}-\lambda\frac{\partial T}{\partial t}\right)\cdot T^{(e)}dV + \int_{S_1^{(e)}} T^{(e)}\cdot q^{(e)}dV+\int_{S_2^{(e)}}\frac{1}{2}\cdot\left[(T^{(e)}-T_b)^2\right]\cdot h^{(e)}dS\right], \quad (7)$$

where $\left\{g^{(e)}\right\}^T=\left[\frac{\partial T^{(e)}}{\partial x}\ \ \frac{\partial T^{(e)}}{\partial y}\right]$, $\left[D^{(e)}\right]=\begin{bmatrix} rK_{rr}^{(e)} & 0\\ 0 & rK_{zz}^{(e)}\end{bmatrix}$ - is the thermal conductivity matrix.

In this case, $r=(R_1+R_2+R_3)/3$ means the distance from the axis of symmetry to the center of the element.

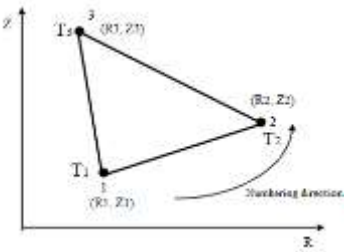


FIGURE 1. Triangular finite element

The temperature change inside the finite element is expressed through the node values:

$$T^{(e)}=N(r,z)T(t)$$

or in matrix form:

$$T^{(e)}=\left[N_1(r,z)N_2(r,z)N_3(r,z)\right]\begin{Bmatrix}T_1(t)\\T_2(t)\\T_3(t)\end{Bmatrix}. \quad (8)$$

$$\left[c^{(e)}\right]\frac{\partial\{T\}}{\partial t}+\left[k^{(e)}\right]\{T\}+\left\{f^{(e)}\right\}=0, \quad (9)$$

where

$$[c^{(e)}]=\int_{V^{(e)}}\rho c[N^{(e)}][N^{(e)}]^T dV, \quad (10)$$

$$[k^{(e)}]=\int_{V^{(e)}}[B^{(e)}][D^{(e)}][B^{(e)}]^T dV+\int_{S_2}h[N^{(e)}][N^{(e)}]^T dS, \quad (11)$$

$$\left\{f^{(e)}\right\}=-\int_{V^e}(rQ)[N^{(e)}]^T dV+\int_{S_1}q[N^{(e)}]^T dS-\int_{S_2}hT_b[N^{(e)}]^T dS, \quad (12)$$

where

$V^{(e)}$ is the volume of the finite element;

$[N^{(e)}]$ is the matrix containing the shape functions;

$[B^{(e)}]$ is the matrix containing the derivatives of the shape functions.

When solving the problem of axisymmetric heat propagation, integral (10) has the following form:

$$[c^e]=\frac{2\pi\rho cA}{180}\begin{bmatrix}c_{11}&c_{12}&c_{13}\\c_{21}&c_{22}&c_{23}\\c_{31}&c_{32}&c_{33}\end{bmatrix}, \quad (13)$$

where

$$c_{11}=12R_1^2+2R_2^2+2R_3^2+6R_1R_2+6R_1R_3+2R_2R_3,$$
$$c_{12}=c_{21}=3R_1^2+3R_2^2+R_3^2+4R_1R_2+2R_1R_3+2R_2R_3,$$
$$c_{13}=c_{31}=3R_1^2+R_2^2+3R_3^2+2R_1R_2+4R_1R_3+2R_2R_3,$$
$$c_{22}=2R_1^2+12R_2^2+2R_3^2+6R_1R_2+2R_1R_3+6R_2R_3,$$
$$c_{23}=c_{32}=R_1^2+3R_2^2+3R_3^2+2R_1R_2+2R_1R_3+4R_2R_3,$$
$$c_{33}=2R_1^2+2R_2^2+12R_3^2+2R_1R_2+6R_1R_3+6R_2R_3.$$

The volume integral $\left[k^{(e)}\right]$ in (14) is calculated using the following formula [13]:

$$[k^{(e)}]=\frac{2\pi rK_{rr}}{4A}\begin{bmatrix}b_1b_1&b_1b_2&b_1b_3\\b_2b_1&b_2b_2&b_2b_3\\b_3b_1&b_3b_2&b_3b_3\end{bmatrix}+\frac{2\pi rK_{zz}}{4A}\begin{bmatrix}c_1c_1&c_1c_2&c_1c_3\\c_2c_1&c_2c_2&c_2c_3\\c_3c_1&c_3c_2&c_3c_3\end{bmatrix}, \quad (14)$$

The second integral in (11), if convection occurs between nodes 2 and 3 of the finite element, has the following form:

$$\int_{S_{23}} h[N]^T [N] dS = \frac{2\pi h L_{2-3}}{12} \begin{bmatrix} 0 & 0 & 0 \\ 0 & (3R_2 + R_3) & (R_2 + R_3) \\ 0 & (R_2 + R_3) & (R_2 + 3R_3) \end{bmatrix}, \quad (15)$$

The surface integral $\{f^{(e)}\}$ has the following form:

$$\int_{S_2} hT_b [N]^T dS = \frac{2\pi L_{2-3} h T_b}{6} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix}. \quad (16)$$

If there is heat loss through convection between nodes 1, 2 or 3, 1, matrix (16) changes [7].

If there are no heat sources or heat exchange in an axisymmetric body and heat flux q is specified between nodes 2 and 3 then the "load vector" of the integral element (12) is:

$$\int_{S_1} q[N]^T dS = \frac{2\pi L_{2-3} q}{6} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix}. \quad (17)$$

In solving the non-stationary heat transfer problem, the system of ordinary differential equations takes the following form:

$$[C]\frac{\partial\{T\}}{\partial t} + [K]\{T\} + \{F\} = 0, \quad (18)$$

where

$$[C] = \sum_e [c^{(e)}]; \quad [K] = \sum_e [k^{(e)}]; \quad [F] = \sum_e \{f^{(e)}\}.$$

Substituting the time derivative with its finite-difference approximation in equation (18), a central-difference method for solving the heat conduction equation [1, 7] is obtained:

$$\left(\frac{2}{\Delta t}[C] + [K]\right)\{T\}^{n+1} = \left(\frac{2}{\Delta t}[C] - [K]\right)\{T\}^n - 2\{F\}^{n+1}. \quad (19)$$

Thus, if the temperature vector $\{T\}^n$ is known at t_n, then the temperature of the axisymmetric body at time $t_{n+1} = t_n + \Delta t$ is determined by the system of equations (19) using the square root method.

4. RESULTS AND DISCUSSION

Problem 1. The non-stationary problem of thermal conduction in a homogeneous cylinder made of steel is considered (Fig. 2). The dimensions of the steel cylinder are $R = 0.7\,m$, inner radius $r = 0.2\,m$, and length $L = 0.5\,m$.

Steel possesses the following thermophysical properties:

$K_{rr} = K_{zz} = 46 W/(m^\circ C)$, $\rho = 7800\,kg/m^3$, $c = 460\,J/(kg^\circ C)$.

Constant temperature $T_h = 100^\circ C$ is maintained on the inner boundary $r = 0.2\,m$ of the cylinder, and on the outer side $r = 0.7m$, there is convection with the ambient temperature $T_c = 20^\circ C$ and heat transfer coefficient $h = 10 W/(K \cdot m^2)$. The initial temperature of the cylinder is $T_0 = 50^\circ C$.

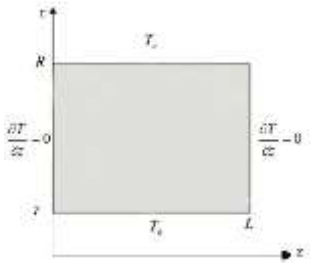


FIGURE 2. Cylinder section and boundary conditions

Figure 3.a shows the general representation of the FE mesh for the cylinder section. Figures 3.b and 3.c show the results obtained with this solution algorithm and the results given in [2].

(a) (b) (c)

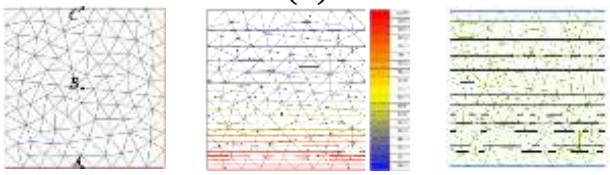


FIGURE 3. a) finite element representation; b) isotherms obtained with solution algorithm; c) isotherms given in [2].

The following characteristics of the FE mesh were used to solve the problem: 192 nodal points, 115 finite elements, the width of the band matrix of non-zero elements of the

system of equations is 20, the final calculation time is $t = 36000\,sec$, and the time step is $\Delta t = 1800\,sec$.

Point (m)	at point A(0.25, 0.2)	at point B(0.25, 0.45)	at point C (0.25, 0.7)
Elcut	100	87.76	82.82
FEM	100	86.52	81.86

TABLE 1. Comparison of numerical values of temperature ($^{\circ}C$)

The numerical solutions are shown to align with the results in [2] by 98%, confirming the accuracy of the developed numerical model and the algorithm used to solve the problem (Table 1).

Problem 2. The problem of heat propagation in an axisymmetric steel structure is considered (Fig. 4.a); it is required to determine the temperature distribution within the body. Constant temperature $T_h = 100\,^{\circ}C$ is maintained on the inner surface of the body, and heat exchange occurs on the outer surface. The ambient temperature is $T_c = 20\,^{\circ}C$ and the heat exchange coefficient is $h = 10 W/(K \cdot m^2)$. The symmetrical cross-section of the body and its dimensions are shown in Fig. 4.b. The initial temperature of the body is $T_0 = 20\,^{\circ}C$. The thermophysical properties of steel are the same as in Problem 1.

Figure 5 shows the body temperature isotherms at different time points (t = 6, 30, and 60 sec) (time step is $\Delta t = 6\,sec$). The heat flow, under the action of temperature gradients, passes from the inner surface of the body to the outer one.

(a) (b)

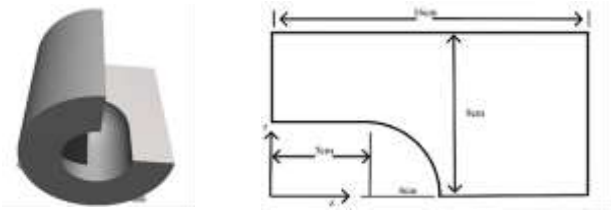


FIGURE 4. Axisymmetric body(a) and symmetric section(b)

To study the effect of the body's geometric characteristics on the temperature field, a problem for a rectangular and circular cylindrical hollow structure is considered. The surface of the void is covered with thermal insulation. To analyze the obtained numerical results, Table 2 shows temperature at different points on the boundary of the hole (at a distance of 3 cm and 5 cm) at various time points.

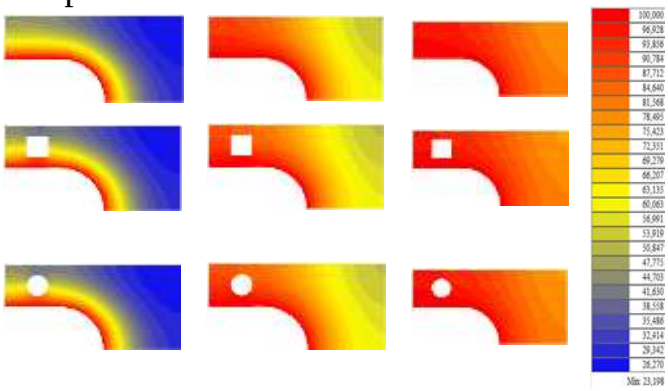


FIGURE 5. Isotherms (at t = 6, 30, and 60 sec)

structure	6 sec	30 sec	60 sec
solid	76,28	95,88	98,63
with square hole	63,42	93,19	97,73
with circular hole	72,12	95,14	98,34

TABLE 2. Temperature values (coordinates of point (3cm; 5cm))

The presence of holes in axisymmetric bodies causes a redistribution of heat flow. In a solid (continuous) structure, heat flow encounters no resistance. However, when cylindrical holes with rectangular and circular cross-sections are present, there is a noticeable change in temperature around these holes. This effect can be observed at the point (3 cm; 5 cm) in Figure 6.

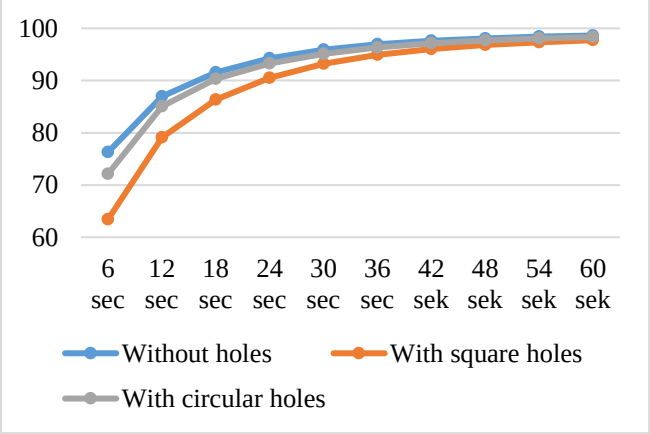


FIGURE 6. Graph of temperature change over time.

4. CONCLUSION

The article introduces a numerical model developed to study the problem of time-dependent heat transfer in axisymmetric bodies. It features a discrete model of a two-dimensional section of a complex structure, accompanied by an algorithm for solving the problem. The study explores the impact of geometric characteristics on the temperature distribution of an axisymmetric body with an inhomogeneous structure. Furthermore, it examines heat flow within a structure containing cylindrical holes with rectangular and circular cross-sections.

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